Seeing the Forest and the Trees When Writing a Mathematical Proof

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Abstract: One of the typical challenges facing a mathematics student when writing a proof is the need to understand the interplay of details and broader concepts. I describe a multi-step proof-writing assignment used in a mid-level course for mathematics majors that is designed to help with this challenge by forcing students to incrementally increase their engagement with the various conceptual levels of the material at hand.

Introduction

Most people walking in a forest can perceive where each of the trees near them is located in relation to the others. They can see any well-cleared paths and can comfortably come to conclusions such as "this path must go up to that hill on the right, and that path must lead over to the stream." They can see the forest, the trees, and more or less the structure of their nearby surroundings.

Mathematics is also a forest, but somehow many young explorers in the realm of mathematics have far less surety, even in their proximate surroundings, than their naturalist counterparts. Developing this surety is one of the key challenges of undergraduate mathematics education. After a final examination I recently gave in an upper-level mathematics course, some students commented to me that the exam "caught them off guard," in that there were problems on the exam for which they were unsure which techniques to bring to bear. In other words, they did not know in which part of the forest they were standing! Sadly, this should not be too surprising. Undergraduate training in mathematics typically puts a heavy emphasis on form-oriented material, often in the form of "problem-solving methods," for example. Students are taught to work with exactingly precise, formally defined terms and symbols, and they are trained in how to properly work with these. As a result, even budding mathematicians may suffer from mathematical myopia.

Professional mathematicians operate differently. Faced with an unfamiliar problem or goal, they set out to build a conceptual framework, to take a good look at the forest around the trees. They may define new terms and prove auxiliary theorems to provide a means for mapping out the conceptual framework of their surroundings. Thus, a major educational goal in a typical Mathematics program is to facilitate students' development of a conceptual framework in which they know "the lay of the land" and they can successfully reason. Unfortunately, the unavoidable need to emphasize formality and technical precision can work against the ultimate goal of developing a mathematically aware, even sophisticated, intellect. From a pedagogical perspective, therefore, the challenge addressed here is how to facilitate students' successful engagement with content while working in a heavily form-oriented context—that they will see both the mathematical forest in which they stand and how it is comprised of the mathematical trees around them.

As a specific case study, I will describe an assignment I have given to students in a course entitled "Linear Algebra for Math Majors." This course is typically taken by second-semester sophomores or first-semester juniors, and an introductory course in mathematical proof-writing is a prerequisite. The following discussion will necessarily involve mathematical terminology, but I will take pains to ensure that the reader can follow without actually understanding the detailed content of the mathematics.

The Prompt and the Issue of Hierarchical Structure

The prompt was as follows:

> Suppose T: V→W is a linear transformation and B is a basis for V. Prove that T is an isomorphism if, and only if, T(B) is a basis for W.

The difficulties students face with a prompt such as this one grow from a deep hierarchical structure in the terms and logical relations between those terms. Students must be able to understand the prompt as having the following structure:

Suppose

T ***is a*** <u>linear transformation from V to W</u>

and

B ***is a*** <u>basis</u> for V.

Prove that

T ***is an*** <u>isomorphism</u>

if, and only if,

T(B) ***is a*** <u>basis</u> for W.

Here, the words in ***bold italic*** are structural, and the underlined terms are units of content. Indeed, some texts, such as Scheinerman (2013), make this kind of deconstruction a pedagogical cornerstone of their approach to introducing students to writing mathematical proofs. Following that view, students are taught to deconstruct the structure further, by expanding each underlined term, potentially producing new underlined terms in the process. In this case, we obtain the following (compare carefully with the above structure!):

Suppose

T ***is a*** function from V to W

such that

for any vectors u,v in V ***and any*** scalars a,b

we have T(au+bv) = aT(u)+bT(v)

and

B ***is a*** set of vectors in V

such that

for any non-zero vector v in V

there exists a unique way to express v as a linear combination of elements of B

Prove that

T ***is a*** function

such that

for any vectors u,v in V,

if T(u) = T(v) ***then*** u = v

and

for any vector w in W,

there exists v in V ***such that*** T(v) = w

if, and only if,

the set of all T(v), ***where*** v is a vector in B,

is a set such that

for any non-zero vector w in W

there exists a unique way to express w as a linear combination of elements of that set

It is possible to go even another layer deeper, by expanding the underlined terms "function," "linear combination," and "vector." The key point here, though, is that the deconstruction process relegates the conceptual content to ever lower and isolated components of the overall structure, while leaving behind a skeleton of logical connectives and other purely structural terms.

This approach, however, can give students the impression that proving theorems is a process of *deconstruction of form* followed by addressing *isolated details*, and moreover it can lead students away from, rather than towards, engaging with the content as a unified entity. My goal, and the goal of assignments such as the one I am highlighting here, is to inculcate a different approach. Now, of course I am not advocating ignoring the logical form of the statement to be proven. This would be a rank foolishness because no mathematical statement can be proven true outside the framework of its logical structure. I argue for a more subtle approach, namely, that proving theorems is a process of *engaging with content* in its formal manifestations, accounting for *details in the context* of the whole.

Incremental Engagement

A typical assignment in a proof-oriented mathematics class would likely consist of just the original two-sentence prompt. Students might be given comments and asked to revise their work, but it would be unlikely there would be further elaboration of the assignment. The general idea of the writing assignment which I am describing here, however, was to incrementally assist students with perceiving the hierarchical structure of the statement they were to prove. While this may be construed as simply leading students through the deconstruction process I wrote against above, there is an important pedagogical point which renders this perception invalid: at each incremental stage, students were required to turn in a complete solution, and their work was graded and received thorough comments. This ensured that at all times they were engaging as best they could with the content and context of the statement to be proven.

I present the assignment in numbered stages.

(*i*) In the first stage, students were provided with just the original prompt, after all of the underlined terms had already been carefully defined and discussed in lectures as well as in the book. Of course, this does not mean that these definitions had yet been properly absorbed. Indeed, the work turned in by the students showed quite clearly that this was, in general, not the case.

(*ii*) Hypothetically, if student work in stage (*i*) indicates it is necessary, a next stage might be to supply students with just the most gross structure of the outline of a proof:

> **Suppose** T is an isomorphism...
>
> ¦
>
> ... **thus we conclude** T(B) is a basis.
>
> **Suppose** T(B) is a basis...
>
> ¦
>
> ... **thus we conclude** T is an isomorphism.

This outline very specifically addresses the logical structure created by the phrase "if, and only if." As it turned out, all of the students in my class successfully dealt with this piece of structure (presumably, in large part due to the training they had received in their prerequisite introduction-to-proof-writing course), so this step was skipped.

(*iii*) Again hypothetically, the next stage would be to explicitly unpack the definitions of the key terms appearing in the outline from stage (*ii*):

> **Suppose** T is an isomorphism,
>
> > ***i.e.,*** T is one-to-one and onto.
> >
> > ¦
> >
> > ... **thus we conclude** that T(B) is a linearly independent set.
> >
> > ¦
> >
> > ... **thus we conclude** that T(B) is a spanning set,
>
> **and thus we conclude** T(B) is a basis.

Suppose T(B) is a basis,

i.e., T(B) is a linearly independent and spanning set.

¦

. . . thus we conclude that T is one-to-one.

¦

. . . thus we conclude that T is onto,

and thus we conclude T is an isomorphism.

In fact, I did not provide this stage to my students, but their work on the next stage made me realize they would have been well served by going through this intermediate stage.

(*iv*) In this stage, the outline of stage (*iii*) is further refined by indicating for each of the spots marked "¦" more specifics about which properties to invoke in working towards the relevant sub-conclusion. Thus, for instance, the first half of the outline would be elaborated as follows:

Suppose T is an isomorphism,

i.e., T is one-to-one and onto.

Because T is one-to-one **we have that . . .**

¦

Because B is a basis **we have that . . .**

¦

. . . thus we conclude that T(B) is a linearly independent set.

Because T is onto **we have that . . .**

¦

Because B is a basis **we have that . . .**

¦

. . . thus we conclude that T(B) is a spanning set,

and thus we conclude T(B) is a basis.

Discussion and Conclusion

Over a series of assignments, it makes sense to progressively give students less and less assistance. Within a single assignment, however, providing too much assistance up front can negatively impact the way students think about the material at hand. Because of the highly modular nature of typical mathematical proofs, if students are presented with a proof skeleton up front, there will be no need nor motivation for them to engage with the subject of the proof in its totality. A proof skeleton for a typical undergraduate-level proof partitions the conceptual landscape into highly restricted and logically isolated chunks. While it may be efficient to deal with each of these individually, such an exercise fails to encourage the kind of holistic thinking towards which the assignment described here is directed.

One key consideration to keep in mind when constructing a "forest and the trees" assignment is that to make sense of one's surroundings, one must have some awareness of them. Students presented with this kind of assignment must have already gained some experience and facility with related material and be at least familiar with the relevant definitions. Approaching stage (*i*), students begin examining their store of conceptions of potentially relevant ideas and considering what will be useful and what will not. Because their perspective has not been narrowed by being given a proof skeleton, they will think broadly, even if haphazardly. In other words, they will engage with the subject of the proof in its totality. Indeed, one student, in reflecting on what he learned from the assignment, said that it helped him develop useful questions to ask himself when writing proofs. When asked to elaborate on this thought, he responded

> I start by ensuring I know where to start and where I will end...
>
> I briefly consider the different assumptions and try to map in my head which one will get me to the conclusion in a clear and concise manner. Sometimes of course, for longer proofs, this doesn't always work and I may make the "wrong" assumption and have to start again.

Another student commented similarly that "the final step wasn't obvious from the first, and you had to think about where exactly you were going."

The additional guidance students receive in stages (*ii*) and on then serves as an impetus for them to reflect on what they should have done, showing them the layout of the terrain, and key paths through it, that they had been examining less efficiently and effectively than is desirable. As a third student commented, the assignment "was incredibly helpful in learning how we should be thinking about the proofs and structuring them." But the cognitive benefit runs deeper. Each time students receive further guidance, in the form of a more refined outline, they are forced to map that increased awareness of the logical structure onto their previous conception of the interconnections of the relevant concepts. This repeated reconsideration of the material, with increased sophistication at each stage, is the key to the benefits of this assignment.

The work the students turned in for stage (*iv*) did indeed show a significantly increased sensitivity to the overall content of the statement in question over the sensitivity demonstrated by their submissions for stage (*i*). On the other hand, the deficiencies in the overall quality of the writing and reasoning, even for stage (*iv*), emphasize that exercises such as this cannot be one-time events, but must necessarily be part of a regular sequence of assignments of this type throughout the semester and across multiple courses. This will surely be well received; each of the students who provided feedback on this assignment included a suggestion that a similar assignment be done earlier in the semester.

With repeated encouragement to lift their heads and make sense of their surroundings, students will be accustomed to, and then adept at, seeing both the forest and the trees.

Assignment

See the Supplementary Files for this article at theprompjournal.com for a PDF facsimile of the original formatting of this assignment.

The prompt is multi-part, each part being given to the students after their response to the previous part has been returned with comments.

(i) Suppose T: V→W is a linear transformation and B is a basis for V. Prove that T is an isomorphism if, and only if, T(B) is a basis for W.

(ii) Suppose T: V→W is a linear transformation and B is a basis for V. Prove that T is an isomorphism if, and only if, T(B) is a basis for W by filling in the gaps in the following structure:

> Suppose T is an isomorphism...
>
> ¦
>
> ...thus we conclude T(B) is a basis.
>
> Suppose T(B) is a basis...
>
> ¦
>
> ... thus we conclude T is an isomorphism.

(iii) Suppose T: V→W is a linear transformation and B is a basis for V. Prove that T is an isomorphism if, and only if, T(B) is a basis for W by filling in the gaps in the following structure:

Suppose T is an isomorphism,

i.e., T is one-to-one and onto.

¦

... thus we conclude that T(B) is a linearly independent set.

¦

... thus we conclude that T(B) is a spanning set,

and thus we conclude T(B) is a basis.

Suppose T(B) is a basis,

i.e., T(B) is a linearly independent and spanning set.

¦

... thus we conclude that T is one-to-one.

¦

... thus we conclude that T is onto,

and thus we conclude T is an isomorphism.

(iv) Suppose T: V→W is a linear transformation and B is a basis for V. Prove that T is an isomorphism if, and only if, T(B) is a basis for W by filling in the gaps in the following structure:

Suppose T is an isomorphism,

i.e., T is one-to-one and onto.

Because T is one-to-one we have that ...

¦

Because B is a basis we have that ...

¦

... thus we conclude that T(B) is a linearly independent set.

Because T is onto we have that . . .

⋮

Because B is a basis we have that . . .

⋮

. . . thus we conclude that T(B) is a spanning set,

and thus we conclude T(B) is a basis.

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References

Scheinerman, E. (2013). *Mathematics, A Discrete Introduction* (3rd ed.). Boston, MA: Brooks-Cole.