The prompt is multi-part, each part being given to the students after their response to the previous part has been returned with comments.

(i) Suppose $T: V \rightarrow W$ is a linear transformation and $B$ is a basis for $V$. Prove that $T$ is an isomorphism if, and only if, $T(B)$ is a basis for $W$.

(ii) Suppose $T: V \rightarrow W$ is a linear transformation and $B$ is a basis for $V$. Prove that $T$ is an isomorphism if, and only if, $T(B)$ is a basis for $W$ by filling in the gaps in the following structure:

Suppose $T$ is an isomorphism…

\[ \text{…thus we conclude } T(B) \text{ is a basis.} \]

Suppose $T(B)$ is a basis…

\[ \text{…thus we conclude } T \text{ is an isomorphism.} \]

(iii) Suppose $T: V \rightarrow W$ is a linear transformation and $B$ is a basis for $V$. Prove that $T$ is an isomorphism if, and only if, $T(B)$ is a basis for $W$ by filling in the gaps in the following structure:

Suppose $T$ is an isomorphism, \( i.e., \) $T$ is one-to-one and onto.

\[ \text{… thus we conclude that } T(B) \text{ is a linearly independent set.} \]

\[ \text{… thus we conclude that } T(B) \text{ is a spanning set,} \]

\[ \text{and thus we conclude } T(B) \text{ is a basis.} \]

Suppose $T(B)$ is a basis, \( i.e., \) $T(B)$ is a linearly independent and spanning set.

\[ \text{… thus we conclude that } T \text{ is one-to-one.} \]

\[ \text{… thus we conclude that } T \text{ is onto,} \]

\[ \text{and thus we conclude } T \text{ is an isomorphism.} \]

(iv) Suppose $T: V \rightarrow W$ is a linear transformation and $B$ is a basis for $V$. Prove that $T$ is an isomorphism if, and only if, $T(B)$ is a basis for $W$ by filling in the gaps in the following structure:

Suppose $T$ is an isomorphism, \( i.e., \) $T$ is one-to-one and onto.

Because $T$ is one-to-one we have that …
Because $B$ is a basis we have that …

… thus we conclude that $T(B)$ is a linearly independent set.

Because $T$ is onto we have that …

Because $B$ is a basis we have that …

… thus we conclude that $T(B)$ is a spanning set, and thus we conclude $T(B)$ is a basis.