The assignments

(All mathematical terminology in the prompts for Day 1 through Day 12 is standard. The reader can find precise definitions in many sources. The terminology in the prompts for Day 13 through Day 15 is not standard, and definitions are supplied just prior to those prompts. An example of a sample response to a prompt is included after the list of assignments. Such sample responses were made available to students just after the corresponding writing work was completed.)

Day 1

1. Pick an odd integer and prove that it is odd.

2. Pick an even integer and prove that it is not odd.

3. Optional challenge: Prove that no integer can be both even and odd.

Day 2

1. Pick a mixed number and show that it is rational.

2. Pick a terminating decimal and show that it is rational.

3. Optional challenge: Prove that $0.27272727\ldots$ is rational.

Day 3

1. Pick a multiple of 12 and show that it is also a multiple of 3.

2. Suppose that x and y are unknown numbers such that 5 divides x and 11 divides y. Prove that 55 must be a divisor of their product xy.

Day 4

1. Prove that the sum of any two even integers must be an even integer.

2. Prove that the square of every multiple of 3 must be a multiple of 9.

Day 5

1. Prove or disprove: every time we multiply two even numbers, the result is a multiple of 4.

2. Prove or disprove: every odd integer from 3 on up is prime.

Day 6

1. Assume that x is an integer. Prove that if x^2 is odd, then x must be odd.

2. Optional challenge: Assume that x and y are integers. Prove that if xy is even, then at least one of x or y must be even.

Day 7

(The universe for this exercise is the set of real numbers.) Prove the following statement: if x^3 is irrational, then x must be irrational.

Day 8

Prove the following statement: the sum of every two rational numbers must be a rational number.

Day 9

On the set of all the integers, define the relation R by: $(x, y) \in R$ if and only if $x + y$ is even.

Determine whether R has each of the properties of reflexivity, symmetry, and transitivity. If it does have all three properties, determine the equivalence classes it creates.

Day 10

On the set of all integers, define the relation R by: $(x, y) \in R$ if and only if $x - y$ is even. Prove that R is reflexive, symmetric, and transitive, and then find the equivalence classes that it creates.

Day 11

Define a function f from the real numbers to the real numbers by the formula $f(x) = 0.5x - 5$.

Prove that this function is one-to-one, and prove that it is onto the set of all real numbers.

Day 12

(The universe for these exercises is the set of integers.)

1. Suppose that x is a multiple of 5, suppose y is a multiple of 6, and let $z = 4x + 10y$. Prove that z is a multiple of 20.

2. Suppose that a divides b and that a divides c. Prove that a divides $(b + c)$.

The non-standard definitions which follow were used to match wording in a particular textbook (Wheeler & Brawner). They are useful for extra practice in learning new terminology and using it in writing proofs.

Definition 1. We call an integer n smooth if and only if there exists an integer k such that $n = 3k$.

Definition 2. We call an integer n rough if and only if there exists an integer j such that $n = 3j + 1$.

Definition 3. We call an integer n abrasive if and only if there exists an integer I such that $n = 3i + 2$.

Day 13

1. Prove that the sum of every two rough integers must be an abrasive integer.

2. Prove that the product of every two abrasive integers must be a rough integer.

Day 14

1. Write a careful element-trace proof of that for sets A and B, $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$. (Informally and in words, prove that "the complement of the intersection is the union of the complements.")

2. Prove that if the integer a is not smooth, then a^2 is not smooth, by using these two cases.

2a. Prove that if a is rough, then a^2 is rough

2b. Prove that if a is abrasive, then a^2 is rough.

Day 15

Write a proof by contradiction for each of the following exercises.

1. If an integer is rough, then it is not abrasive.

2. If an integer is rough, then it is not smooth.

3. If an integer is abrasive, then it is not smooth.

Sample responses

For the reader who wishes to see more regarding the goals for these exercises, we include the sample responses that were provided for the Day 12 prompts above.

1. (Suppose that x is a multiple of 5, suppose y is a multiple of 6, and let $z = 4x + 10y$. Prove that z is a multiple of 20.)

Based on the definition of "multiple," we must show that there exists an integer w such that $z = 20w$.

Since x is a multiple of 5 and y is a multiple of 6, we know that there exist integers u and v such that $x = 5u$ and $y = 6v$. Then

$$z = 4(5u) + 10(6v) = 20u + 60v = 20(u + 3v).$$

Now let $w = u + 3v$. Since u and v are integers, w is also an integer, and $z = 20w$, as required.

2. (Suppose that a divides b and that a divides c. Prove that a divides $(b + c)$.)

Based on the definition of "divides," we must show that $a \neq 0$ and that there exists an integer d such that $b + c = da$.

Since a divides b and that a divides c, we know that $a \neq 0$ and that there exist integers e and f such that $b = ea$ and $c = fa$. Then

$$b + c = ea + fa = (e + f)a.$$

Now let $d = e + f$. Since e and f are integers, d is also an integer. Also, we already know that $a \neq 0$, and now we have that $b + c = da$, completing the proof.