# Scaffolded Daily Writing Assignments Introducing the Writing of Mathematical Proofs

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*Abstract*: Writing mathematical proofs is a key component of writing in the discipline in mathematics. Historically, many students have struggled in pursuing this endeavor, particularly during their early exposure to the process. To help students progress toward the goal of being able to consistently create well-written proofs, I present an incremental approach used in a course for elementary education majors who are concentrating in mathematics. This approach uses daily low-stakes writing assignments. Using this instructional technique, I found that student engagement improved and that, overall, better mathematical proofs were written. One more instructor at my institution has already adopted the same methods, and I expect more to do so.

## Introduction

Writing across the curriculum is sometimes addressed too superficially in mathematics classes. For example, a mathematics course may sometimes include one or more assignments such as a short paper about a particular person or event in the field. While this writing exercise is useful, it is not truly in the discipline of mathematics. A paper of this sort might more properly be described as a piece of journalism, a historical article, or something else. Writing in the discipline of mathematics is likely to fall into one of three categories (Flesher, 2003; Russek, 1998). The first of these categories is explanatory, in which the writer wishes to communicate a concept or result to a reader who is not well-versed in the technicalities of the topic at hand. The second is process-oriented, in which the writer details the reasoning throughout an analysis of a particular problem. This category is essentially an expanded version of the familiar instruction to "show your

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work," and it serves both as writing-to-learn (by helping the writer to analyze their own problem-solving process) and as evidence to the reader that the writer has mastered the problem at hand. The final category is formal mathematical proofs, detailed logical arguments which could be said to be the mathematician's version of persuasive essays. Certainly, there is some overlap among these categories; for example, a proof may contain some exposition, or a problem-solving approach may cite various technical theorems. It is fair to say, though, that most mathematical writing has its thrust in one of those three directions.

In this article, the writing of mathematical proofs will be addressed. Many years ago, this skill was usually handled in something of a sink-or-swim fashion. Students in college mathematics classes, particularly in those courses within a mathematics major, were required to read and produce detailed proofs with little instruction on how to build the ability to do so. It was assumed that many technical definitions would be memorized rapidly and used precisely. It was further expected that students would work on their own to understand the broad structure of proofs, including the various categories of logical arguments that are employed—direct proof, contraposition, contradiction, analysis of cases—even though the examples they saw consisted entirely of details. Most teachers rarely discussed a bigger picture, including fundamental underlying concepts and generalizations to other situations, although a few classes would find themselves fortunate enough to be with a professor who dedicated extra effort and time to helping them build an understanding of the art of proofs. The problem is exacerbated when the first proof-intensive course is at a new level of abstraction for students, as is often the case in college classes like linear algebra or abstract algebra (Weber, 2001, 2002). While some institutions have continued along traditional lines, most have begun including significant time, and even entire courses, to the reading and writing of proofs. The positive effects of designing a course specifically for the purpose of introducing proofs were first noted some time ago (Marty, 1986, 1991), and many mathematics departments now offer such a class within their major. The textbooks and papers which emphasize this approach are too numerous to allow an exhaustive list here, but we note Solow (2013), Abrams (2016), and Velleman (2006) as examples.

In a Fall 2016 course in which proofs are introduced, I proceeded more slowly than I had in the preceding several years, beginning with extremely simple concepts for which proofs consisted of nothing more than a line or two showing that a definition is satisfied. Each day, a little more information was incorporated, and the proofs gradually became more involved. I made examples, templates, outlines, or checklists, and I supplied input during and after the in-class writing. Over time, the value of this more incremental approach to teaching proofs evidenced itself.

## Background

The assignments I discuss are used in a mathematics course in a large state college, a course whose student population is future elementary and middle school teachers who are planning to be mathematics specialists. The course title is "Formalizing Mathematical Thought," or "FMT." These are not traditional mathematics majors *per se*. These students, on their way to state-approved teacher certification, cover a full curriculum of courses for elementary educators, and they also take several content courses in mathematics. While their curriculum does not include as much mathematics as that of a B.A. in mathematics, it has always contained courses in calculus, statistics, Polya's problem-solving method, number theory, college geometry, and discrete mathematics. The last three of those courses, as taught at Rhode Island College (RIC), all involve a nontrivial amount of proof work. This includes direct proofs, indirect proofs, and mathematical induction.

The FMT course has used a standard textbook (Wheeler & Brawner, 2010), and an outline is as follows. The coverage includes a chapter on elementary logic, a chapter on set theory, a chapter introducing several techniques of proof and a very brief introduction to number theory, and a chapter on some basic combinatorics. The instructor has the discretion to include other topics as time allows; typically, some or all of a chapter on elementary graph theory is covered. A goal of the course, in addition to the understanding of the assorted content material, is careful explanation of all work and the writing of simple proofs. As the course was taught for its first several years of existence, my colleagues and I were not covering or assigning proofs early in the semester. Instead, we followed the scheme which appears in this textbook (and many others), that of delaying such coverage until some logic and set terminology was in place. Prior to about 2009, instructors would cover the necessary proof techniques in an ad hoc manner as the need arose in the number theory, college geometry, and discrete mathematics courses mentioned above. As a department, we chose to create a new course to be taken earlier in the curriculum, a course whose goals included the reading and writing of proofs. The underlying idea is to use the same strategy that many schools have employed within the mathematics major, that of concentrated exposure and training in proofs themselves rather than having techniques covered here and there as they arise in various contexts. The writing assignments in this article are part of a change in our pedagogical approach.

RIC includes many of its state's future teachers among its thousands of undergraduates. The institutional approach regarding mathematics for education majors has been that while mathematics specialists in elementary schools and middle schools need not have as much technical background as students attaining a traditional bachelor's degree in mathematics (that is to say, topics such as ring theory and multivariable calculus are not required in their curriculum), these students should be strong in foundations closely related to things they will be teaching. This institutional belief is consistent with state teacher certification guidelines. A main goal of the course addressed herein is to give students a solid background in topics related to the reasoning, reading, and writing of proofs in mathematics. That background is intended both to help them in the later, proof-dependent courses noted above, and to deepen their understanding of topics that will come up in their teaching careers.[1]

For years, the FMT course was taught in a format that is common to many college mathematics classes. A typical 110-minute meeting would begin with a conversation about, and student presentation of, some homework problems. Following that, the instructor would lead a presentation and discussion of new material, and, toward the end of the period, the class would be organized into small groups for an in-class assignment, usually one similar to homework problems or the new examples from that day's class. Most students would be doing well in the class for several weeks, as the coverage of elementary logic and set theory proceeded. Then, about a third of the way into the semester, the coverage of the chapter on mathematical proofs would begin, and a significant portion of the class would suddenly begin to struggle. We would make a somewhat quick transition from writing no proofs to writing one proof after another. Quite a few definitions had to be learned in a short time, and an overview of the techniques of writing proofs, starting from simple direct proofs and moving through indirect arguments and mathematical induction, would be covered over a span of a few weeks. There is a considerable amount to be absorbed when it comes to the writing of proofs. Students need to learn that there should be a clear statement of precisely what is going to be shown and why, that any definitions should be cited, that proper grammar must be used, that any variables need to be defined, and so on. For many students, this is quite a bit to absorb. The typical outcome of the portion of the course devoted to introducing proofs would be that about a third of the students would continue to thrive, a third would struggle to partial success, and a third would encounter serious difficulties. The rest of the term, which would include some elementary number theory, basic combinatorics, and usually some graph theory, with a significant emphasis on proofs throughout, would see the class continue in its stratified state.

I taught the course several times during the first six years that it was offered. The atmosphere and results were typical: almost all the students passed, and a few did very well, but quite a few who completed the class did so with what I assessed as merely a satisfactory background in proofs rather than a truly strong foundation. During the 2014–2015 and 2015–2016 academic years, I served on our college's Writing Board, as a representative from mathematics and the sciences. I have always been interested in how best to teach the writing of proofs, and that exposure to the world of writing instruction informed my thinking in a new way. I began to think about proofs as being one example of writing in a discipline. In the Fall 2016 semester, I taught the course differently, with an emphasis on proofs as a type of writing rather than as something peculiar to mathematics alone, using some techniques from the world of writing instruction.

## Summary of Approach

The approach that was used beginning in the Fall 2016 section of this course was based on introducing proof-writing from the first day of classes, using the technique of scaffolding, an approach familiar in writing instruction. Regarding scaffolding in general, Gibbons (2015) describes it as not just any sort of assistance:

> It is a special kind of help that assists learners in moving toward new skills,

> concepts, or levels of understanding. Scaffolding is thus the temporary assistance by which a teacher helps a learner know how to do something so that the learner will later be able to complete a similar task alone. It is future-oriented and aimed at increasing a learner's autonomy. (p. 16)

In designing scaffolded assignments, an instructor initially provides a great deal of support while assigning tasks that are relatively easy, then gradually withdraws the support while making the tasks more complex. In mathematical writing, the support often takes the form of detailed templates, even fill-in-the-blanks types of tasks like providing the justification for each line of a logical argument. Such templates might gradually become outlines, then checklists, and then less-detailed reminders.

The assignments described in this article begin at a low level and slowly increase in difficulty to something more typical for a college proof-based course. The daily in-class preparation and classroom support matched this pattern. At the start of each class period, immediately prior to the day's writing assignment, I began with a proof or two similar to those about to be assigned. I used the common mathematics classroom format of the instructor steering the process but striving for interaction with the class. I gave commentary along the lines of, "Please tell me the definition of this term ... okay, let's write down the key goal here ... have we shown what we needed yet ... " and so on. Every day throughout the term, I emphasized the most important cognitive and writing steps for all proofs: write a clear statement of what needs to be shown, outline the plan of how to show it, be sure to have a summary sentence at the end, and so forth. The support provided to the students during their work gradually tapered. On the first day, completed proofs were left visible to the class while they worked. As a result, the students' first, and modest, writing task was essentially to reproduce some existing proofs with the numbers changed. Within two weeks, I stopped leaving full proofs on the board, instead putting up a summary of the key steps that we had used moments ago. After about two more weeks, I left the class a brief outline rather than a summary, and two more weeks after that, I left only reminder questions on the board. For example, I might leave visible questions like, "Are any variables clearly defined?" or "Did you cite definitions?" I also roamed the room during every assignment, checking in with all students as they worked, and continued to do so for the entire semester.

To set the mood for the course, I led the class through the first simple definitions, two carefully-written example proofs, and an in-class writing prompt, all before the course syllabus was even reviewed when the class met on the opening day of the semester. This was to further my goal of making mathematical writing a natural part of the thinking process for every task in the course. Every succeeding day began with one or more examples of proofs, including a discussion of the principles and definitions involved, followed by a writing prompt: one or more proofs. I made the examples and assignments incrementally more involved as the days went by, and I also worked to incorporate material we encountered as we moved forward through the text. For example, when we covered relations and functions during the set theory chapter of the text, I was sure to have the next few proof prompts involve those topics. The prompts were undertaken by pairs of students, with each reviewing the other's writing as it progressed. The assignments were low-stakes, with no letter or numerical grades assigned, and the emphasis on feedback and revision. Students got rapid feedback; blessed with a small class, I was able to email comments to students within a day, and often within hours. Suggested possible responses to the writing assignments were posted online immediately after class meetings.

Proofs are not rare in college mathematics courses, so it is worth emphasizing how this approach represents an attempt to make a change in pedagogy. I have mentioned that most proof-based courses make a fairly rapid transition from background material to writing proof after proof. My approach was to try to spread out the learning of the mechanics of proof-writing over eight or nine weeks. I wanted to get students into an early and careful habit of writing precise sentences about what they needed to show, how they were going to show it, why their work did indeed show it, and I wanted to establish this foundation before they ran into conceptually difficult mathematics. Then, when the level of the mathematics itself became harder, the fundamentals of how to structure and write a proof were already in place, and many of the technical definitions were already second nature.

The new approach was almost as though I taught two courses in parallel for the first several weeks of the semester: one course gradually building the writing of proofs, and the other covering the textbook's introductory topics in its traditional order. The time spent on the in-class writing assignments replaced some of the time that had previously been devoted to routine small-group tasks on problems similar to homework and examples from the day's lecture portion. There was no loss of coverage for the semester, in part because of this reallocation of time, and in part because the textbook chapter in which proofs are introduced could now be covered more efficiently than in the past. Whereas we had sometimes spent as many as five or even six weeks on the proof chapter in the past, this time the class was comfortable getting through it in three and a half weeks. The topics of direct proof and proof by contraposition had been thoroughly introduced, so we could cover the corresponding textbook sections reasonably quickly when the time came, and the applications to number theory went smoothly. We also were able to spend sufficient time on the difficult topic of mathematical induction without worrying about having to rush coverage later in the term.

## How the Course Went

The approach was a success. Small class sizes the last few years mean that I cannot obtain meaningful statistical inferences from the higher grades achieved in Fall 2016, but I can still be informed by the atmosphere in the classroom, the experiences of certain students, and the student comments at the end of the semester.

At the end of each semester, our department has every student in every course complete an evaluation regarding the course, the text, and the instructor. This includes numerical ratings intended to represent the student's opinion about the effectiveness of each of those factors, and they also include opportunities for open-ended comment. Student ratings were higher this term than they were in the past, although again we cannot make rigorous conclusions due to small sample sizes. Some example items are of interest, though. In past terms, the open-ended comments usually included about a quarter of the class saying words to the effect that "the proofs lost me." This time, there were no such comments at all, and several papers mirrored this particular quote: "The short proof assignments everyday [*sic*] were a big help, so that when we got to the chapter on proofs, we already had practiced a lot of it." Several more students mentioned that the daily proof prompts helped foster a good classroom environment, with one noting, "It's good to have a little challenge every class but without having to worry that it's going to wreck your grade because you might get a low score like on a quiz. And it's fun talking about the proofs with your partner."

Throughout the semester, I had increased traffic at my office hours. It is often difficult to get many students to come to an instructor's office to explore questions; this time around, though, I had several different students come to discuss my comments on their proof-writing work. It would not surprise me if this was partly as a result of the combination of rapid and frequent feedback and the fact that the assignments had commentary but no letter or numerical grade. I believe that this term, a number of students were invited into a discussion with me rather than just deciding that "B" or "17/20" was good enough for today and stuffing a quiz into a folder.

Two students were repeating the course when they took it in Fall 2016. One had earned an F a year before, and the other had a D in that same term. There are many reasons for low grades, of course, but it is worth noting that both of these students had had a satisfactory beginning the first time through, then wilted when they were unable to quickly synthesize the material in the chapter on proofs when we reached it several weeks into the term. This time through, they responded well to the gradual exposure to proofs, and they asked more questions and volunteered more answers every day. Both saw dramatic improvement in their level of success, and their term grades ended up being B-minus and B-plus, respectively.

In Fall 2017, the course was taught by a different instructor, and it was her first time teaching this class. She chose to implement the approach and proof-writing prompts described herein. She noted that there was a better atmosphere in class than she expected, and, at the end of the term, the student evaluations in the class included a startling unanimous "5 out of 5" rating in "overall teaching effectiveness." She reported that she believed that the design and use of the writing prompts were significant factors in student satisfaction and achievement.

It seems fair to draw the inference that the approach that was taken, that of integrating the writing of proofs into the daily work of the class from the very start, helped to cement proof-writing as an integral part of all the course work rather than just another one-chapter topic among several others.

### About the Writing Prompts

The writing prompts are organized in an incremental structure, starting at a very modest level, with each day's work being slightly more involved than previous assignments. For example, the first day's work involves only the definitions of even and odd integers. No variables appear until the third assignment, and students do not have to introduce variables of their own until the fourth day. Indirect arguments follow a little later, and more complicated proofs, ones with multiple parts, first appear around the fifth week of assignments. I note again that each day begins with some instruction regarding proofs, including one or two examples of proofs that would be somewhat similar to the ones the students were about to attempt. I also emphasize that students have support from completed work, outlines, or reminders left visible to the whole class, and that the instructor supplies advice to the pairs of students as they work.

The assignments here cover about two-thirds of a semester. The class had 28 class meetings scheduled for the semester. Three were used for testing, one was devoted to review for the final exam, and one class was canceled due to a power outage, so there were 23 meetings in which substantial new material was covered. By the last third of the semester, the homework and lecture examples themselves contained many proofs. As a result, I did not need to supply new proof-writing prompts each day, and we spent that writing time at the start of class sharing or improving the work that students had been doing on problems from homework and assigned reading.

### Conclusion

The writing of proofs is a crucial element in writing in the discipline of mathematics. Learning how to consistently produce a good product is a challenge for many students. Writing instructors have long known that certain techniques—including frequent low-stakes writing, clear expectations, peer review, rapid feedback, and a scaffolded approach—improve results and engagement in writing-intensive courses. While it takes some effort for a mathematics instructor to first employ these tools while teaching the writing of proofs, that effort is a valuable investment. The emphasis on starting to learn the style of proof-writing commencing on the first day, and continuing in every class meeting, meant that there was much less of a shock when those topics arrived in a torrent some weeks into the term. It was possible for students who would struggle with writing style or with memorizing definitions to recognize their issues early in the term, meaning that they had time to ask questions and start coming to office hours in time to get the assistance that they needed. Students expressed more confidence and enjoyment, and they appeared to have an improved background for their higher-level courses and for their professions. Finally, the successful experience has given us valuable evidence to use in working to convince colleagues to employ some of the same techniques from writing instruction in a variety of service courses in mathematics.

## Assignment

*See the Supplementary Files for this article at thepromptjournal.com for a PDF facsimile of the original formatting of this assignment.*

(All mathematical terminology in the prompts for Day 1 through Day 12 is standard. The reader can find precise definitions in many sources. The terminology in the prompts for Day 13 through Day 15 is not standard, and definitions are supplied just prior to those prompts. An example of a sample response to a prompt is included after the list of assignments. Such sample responses were made available to students just after the corresponding writing work was completed.)

Day 1

1. Pick an odd integer and prove that it is odd.

2. Pick an even integer and prove that it is not odd.

3. Optional challenge: Prove that no integer can be both even and odd.

Day 2

1. Pick a mixed number and show that it is rational.

2. Pick a terminating decimal and show that it is rational.

3. Optional challenge: Prove that $0.27272727\ldots$ is rational.

Day 3

1. Pick a multiple of 12 and show that it is also a multiple of 3.

2. Suppose that $x$ and $y$ are unknown numbers such that 5 divides $x$ and 11 divides $y$. Prove that 55 must be a divisor of their product $xy$.

Day 4

1. Prove that the sum of any two even integers must be an even integer.

2. Prove that the square of every multiple of 3 must be a multiple of 9.

Day 5

1. Prove or disprove: every time we multiply two even numbers, the result is a multiple of 4.

2. Prove or disprove: every odd integer from 3 on up is prime.

Day 6

1. Assume that $x$ is an integer. Prove that if $x^2$ is odd, then $x$ must be odd.

2. Optional challenge: Assume that $x$ and $y$ are integers. Prove that if $xy$ is even, then at least one of $x$ or $y$ must be even.

Day 7

(The universe for this exercise is the set of real numbers.) Prove the following statement: if $x^3$ is irrational, then $x$ must be irrational.

Day 8

Prove the following statement: the sum of every two rational numbers must be a rational number.

Day 9

On the set of all the integers, define the relation $R$ by: $(x, y) \epsilon R$ if and only if $x + y$ is even. Determine whether $R$ has each of the properties of reflexivity, symmetry, and transitivity. If it does have all three properties, determine the equivalence classes it creates.

Day 10

On the set of all integers, define the relation $R$ by: $(x, y) \epsilon R$ if and only if $x - y$ is even. Prove that $R$ is reflexive, symmetric, and transitive, and then find the equivalence classes that it creates.

Day 11

Define a function $f$ from the real numbers to the real numbers by the formula $f(x) = 0.5x - 5$. Prove that this function is one-to-one, and prove that it is onto the set of all real numbers.

Day 12

(The universe for these exercises is the set of integers.)

1. Suppose that $x$ is a multiple of 5, suppose that $y$ is a multiple of 6, and let $z = 4x + 10y$. Prove that $z$ is a multiple of 20.

2. Suppose that $a$ divides $b$ and that $a$ divides $c$. Prove that $a$ divides $(b + c)$.

The non-standard definitions which follow were used to match wording in a particular textbook (Wheeler & Brawner). They are useful for extra practice in learning new terminology and using it in writing proofs.

Definition 1. We call an integer $n$ *smooth* if and only if there exists an integer $k$ such that $n = 3k$.

Definition 2. We call an integer $n$ *rough* if and only if there exists an integer $j$ such that $n = 3j + 1$.

Definition 3. We call an integer $n$ *abrasive* if and only if there exists an integer $i$ such that
$n = 3i + 2$.

Day 13

1. Prove that the sum of every two rough integers must be an abrasive integer.

2. Prove that the product of every two abrasive integers must be a rough integer.

Day 14

1. Write a careful element-trace proof of that for sets $A$ and $B$, $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$. (Informally and in words, prove that "the complement of the intersection is the union of the complements.")

2. Prove that if the integer $a$ is not smooth, then $a^2$ is not smooth, by using these two cases.

2a. Prove that if $a$ is rough, then $a^2$ is rough.

2b. Prove that if $a$ is abrasive, then $a^2$ is rough.

Day 15

Write a proof by contradiction for each of the following exercises.

1. If an integer is rough, then it is not abrasive.

2. If an integer is rough, then it is not smooth.

3. If an integer is abrasive, then it is not smooth.

## Sample Responses

For the reader who wishes to see more regarding the goals for these exercises, we include the sample responses that were provided for the Day 12 prompts above.

1. (Suppose that $x$ is a multiple of 5, suppose that $y$ is a multiple of 6, and let $z = 4x+10y$. Prove that $z$ is a multiple of 20.)

Based on the definition of "multiple," we must show that there exists an integer $w$ such that $z = 20w$.

Since $x$ is a multiple of 5 and $y$ is a multiple of 6, we know that there exist integers $u$ and $v$ such that $x = 5u$ and $y = 6v$. Then

$$z = 4(5u) + 10(6v) = 20u + 60v = 20(u + 3v).$$

Now let $w = u + 3v$. Since $u$ and $v$ are integers, $w$ is also an integer, and $z = 20w$, as required.

2. (Suppose that $a$ divides $b$ and that $a$ divides $c$. Prove that $a$ divides $(b + c)$.)

Based on the definition of "divides," we must show that $a \neq 0$ and that there exists an integer $d$ such that $b + c = da$.

Since $a$ divides $b$ and that $a$ divides $c$, we know that $a \neq 0$ and that there exist integers $e$ and $f$ such that $b = ea$ and $c = fa$. Then

$$b + c \; = \; ea + fa \; = \; (e + f)\, a.$$

Now let $d = e + f$. Since $e$ and $f$ are integers, $d$ is also an integer. Also, we already know that $a \neq 0$, and now we have that $b + c = da$, completing the proof.

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## Notes

[1] As a simple example, a school mathematics specialist organizing a curriculum that includes the sum-of-digits test for divisibility by three or nine should have a thorough understanding of why those tests are valid.

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