The "this feels like magic" error: Some students have difficulty with inductive proofs because they do not believe something so mechanical could possibly work to prove substantive claims. These students master the four steps of writing a proof by induction, but find it too easy to be trusted. Typically the student's response will be to try to use a non-inductive approach, which will turn out to be (depending on the problem) either much more difficult or in fact impossible.

3. Why Analogies?

At its core, induction is actually a very simple idea. One could faithfully gloss the three inductive statements as follows:

PWI. *If you can get started, and you can keep going, you can get as far as you want.*

PSI. *If you can get started, and (possibly relying on your momentum) you can keep going, you can get as far as you want.*

WOP. *If something happens, but did not always happen, it must have* begun *to happen.*

In my own experience—as a student, in discussion with colleagues, reviewing textbooks—induction is nearly always informally introduced by means of one of the following analogies:

dominoes: The base case is the first domino to topple; the inductive step is when one domino falls onto the next.
walking: The base case is the first step; the inductive step is the act of taking a step.
climbing stairs or a ladder: The base case is the first step or rung; the inductive step is the act of using one stair or rung to get to the next.

Typical practice is to introduce these analogies when induction is first introduced in class, before the formal aspects of an inductive proof, and then shift gears into the formal aspects.

While these analogies serve as a useful introduction, most students discard them as mere dicta. The standard mistakes articulated above can all be minded and mended, or at least mitigated, with reference to such analogies. The main difficulty, then, is to get the student to invest in and refer to the analogy. To this end, I built on Holden's original analogy assignment (2015) by extending it both in time and in depth.

My goals for this assignment are for students to:

- think deeply about each part of a proof by induction
- contextualize the abstract PWI, PSI, and WOP statements into their lived experience
- distinguish among PWI, PSI, and WOP, while making in-context connections among them
- critically evaluate each others' work
- accept PWI, PSI, and WOP as natural outgrowths of ordinary reasoning

4. The Assignment

I will describe the details of the assignment, as it currently stands, having been refined over the course of four semesters' use. As this is a write-to-learn assignment, the precise formatting and style of the final product are not specified or evaluated. In this decision, and many of the choices in structuring the assignment, I was greatly informed by Bahls's work (Bahls, 2012).

4.1. Holden's Original Assignment.

Holden has his students select an analogy topic, write their analogy, and share it with the class (2015). Holden emphasizes, both to his colleagues and in the assignment itself, that the analogy for induction is "merely an analogy", i.e. the students are not producing a true proof by induction. His assignment is also small in scale: it is a one-class activity which asks students to address only PWI. I extended this idea as described below.

4.2. Scope.

In all, the induction analogy assignment runs the entire length of the induction unit—about three weeks.

4.3. **Analogy Selection.** I leave selection of the analogy itself to the students. Here are the instructions:

- Pick a word which starts with the same letter as your family name.
- This word cannot be *count*, *stairs*, *ladder*, *dominoes*, *walking*, or any of the other analogies for induction that we already have.
- Your word can also not be obviously derivative of any of the analogies for induction that we already have (for example, since we've used *walking*, you can't use *run*.)

The last two points are essential: some students will inevitably try to take the easy way out by adapting an existing analogy, which defeats the entire exercise. The purpose of the first point is two-fold. First, it prevents different students picking the same analogy. Second, the constraint forces students to be a little more creative (Stokes, 2005); they cannot pick the first analogy that came to their head and must search around a bit for real-world situations where the inductive idea is present. (It is important to be somewhat flexible, as students whose family names begin with rare-in-English letters like Q, X, Z, etc., may have some difficulty finding a word.)

I collect analogy proposals and approve or require a change within two or three days of the initial assignment. About two-thirds of initial submissions are approved. (I like to approve them with a positive comment such as, "This sounds like it has real promise!") Some submissions I reject as derivative of already-discussed analogies. Others strike me as unworkable, which prompts a back-and-forth with the student to clarify what their analogy will be. Sometimes this back-and-forth results in the student choosing another word; sometimes it turns out they have just been much more creative than I had thought!

4.4. **Initial Analogy (PWI).** The instructions for the initial submission of the analogy are as follows:

> Write an explanation of induction using your word as an analogy. This analogy must explicitly address each aspect of a proper proof by induction:
> (1) the base case $P(1)$
> (2) the inductive hypothesis $P(n)$
> (3) the inductive step *for all* n, $P(n) \Rightarrow P(n+1)$
> (4) the conclusion *for all* n, $P(n)$
>
> Your write-up must also include some kind of visual depiction. You can draw or create the depiction yourself. You can find it somewhere (in a book, online, etc.). You must credit your source. Your depiction must illustrate all the parts of your analogy.

4.5. **Peer Feedback.** Following the submission (with a break of a day or so), I assign student groups of three to four for peer critique and feedback. The peer feedback instructions ask the evaluators to assess, in a paragraph or two, the *clarity*, *accuracy*, *relevance*, *depth*, *originality*, and *appropriateness* of the peer's submission.[1] I also provide the following guiding questions for peer evaluation:

- Does the analogy feature each of the four parts of a proper proof by induction?
- Is the write-up true to the nonmathematical subject of the analogy?
- Is there a way to make the analogy more forceful?
- Does the explanation of the inductive step show how $P(n+1)$ depends on or develops from $P(n)$?

In addition to peer feedback, I provide feedback of my own to each student.

4.6. **Revision and Extension.** For a grade of "B", students must revise their submission according to my feedback and that from their peer group. For a grade of "A", students must, as part of their revision, extend their analogy to encompass the PSI; that is, they must explain how their analogue of $P(n+1)$ depends not merely on $P(n)$, but also on $P(n-1)$, $P(n-2)$, etc.

4.7. **Reflection.** Each stage of the assignment is accompanied by questions in the course reflection journal:

- Describe your process of coming up with a word to use for your analogy. Were there any words you considered, but rejected? Why? What criteria did you use in selecting your word?
- Among the Intellectual Standards of Critical and Creative Thinking, pick three which might be appropriate to evaluate your work on this assignment.
- Of your peers' analogies, which did you find most surprising?

[1]These qualities are drawn from a list of thirteen *standards of critical and creative thinking* which are a running theme in the course (Cooper, 2018). They are adapted from Paul and Elder (2012).

- Pick one member of your peer group, and answer the following: given your peer's letter, could you come up with an analogy for induction? You don't have to do the whole assignment over again, just state the analogy.

4.8. **Administration.** The entire assignment is administered via my institution's Google Docs/Google Drive. This is helpful in checking and recording student progress in all aspects of the assignment, but particularly in coordinating peer feedback. The comment feature also allows for back-and-forth discussion within peer groups; this happened spontaneously in some instances, and I plan to encourage it in future uses of the assignment.

5. Some Examples of Student Work

5.1. **Student Analogies.** While some of the student responses have been only mediocre, I have been quite impressed with the creativity, depth, and attention to detail some students have shown in their work on this assignment. Some salient examples, which I think speak for themselves (analogies are stated in my own words unless expressly quoted):

hammer: The base case is setting the nail; the inductive step is the somewhat different action of using the rebound from one strike to power the next.

phalanx: A phalanx is a military formation of spear-and-shield armed soldiers. Each phalangite carries a shield over the left arm and uses the right arm to fight with a long pike. The inductive content of the analogy that each phalangite ($P(n)$) provides protection to the soldier to his left ($P(n+1)$), via the large shield. This leaves the phalangite at the far right of the line ($P(1)$) in a uniquely vulnerable position; experienced soldiers were placed there to anchor the unit.

gallop: I nearly disallowed this analogy as too close to *walk*, but I ultimately allowed it because the student demonstrated an emerging understanding of PSI. My faith was rewarded in the extension part of the assignment:

> "For a horse to have made 4 galloping motions, the horse must have made 3 previous motions being the 3rd motion, 2nd motion, and 1st motion."

This was accompanied by an illustration of the sequence of four 'motions' that make up a gallop. My only suggestion to this student was to amend "to have made 4 galloping motions" to "to make the 4th galloping motion".

axon: "The inductive step $P(n) \Rightarrow P(n+1)$ is the step where the signal in one node $P(n)$ causes a change in charge of the next node $P(n+1)$, causing the node $P(n+1)$ to depolarize and thus causing the signal to continue into node $n+1$."

5.2. **Student Feedback and Reflection.** Initially, many submissions reflect a surface-level understanding of induction as a *sequence* of events. Through the peer feedback process (both in giving and in receiving peer feedback), and in reflecting on their own process and work, most students come to a deeper understanding of the inductive idea that the successive predicate $P(n+1)$ depends on prior predicates (in the case of weak induction, just $P(n)$; in the case of strong induction, perhaps other prior predicates also). Here are some examples of peer feedback:

> "I think the analogy could benefit from specifying units or layers just to give a bit more conceptual context. In this particular example, it looks like layers of snow would be the most appropriate here. (So, it would just be a matter of changing "n" to "n layers"). However, I think the analogy is otherwise easy to follow and is logically sound. It also has a solid foundation for expanding the analogy to complete [strong] induction."

> "Is a one story building a skyscraper? . . . . I would recommend a different picture for the $n+1$ step because it includes a number of unfinished floors. It is better to clearly outline which part is the n assumption and which part is the $n+1$."

The reflection questions also generated some interesting responses:

> **Among the Intellectual Standards of Critical and Creative Thinking, pick three which might be appropriate to evaluate your work.** "I feel like three words that might be appropriate to evaluate my work on this assignment is precision, depth, and logic. Precision because for this assignment, I would need to be precise on what I am saying on this