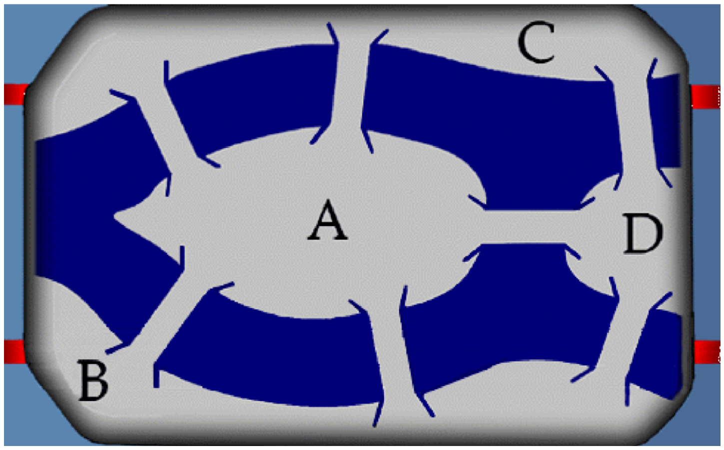
The Konigsberg Bridge Problem

Problem 1: Consider the following layout in K\"{o}nigsberg, Prussia:



Is there a route through the city that crosses each bridge *exactly* once?

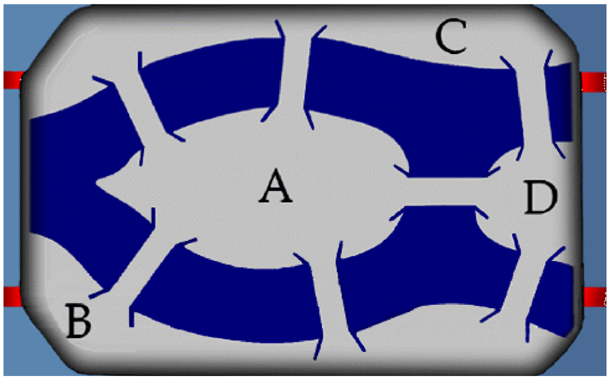
There are two ways we can try to figure out the answer:

1. If we think such a route exists, then ___we should find an example_________________.
2. If we think such a route does not exist, then __we must prove it____________________.

Which method is easier? _#1_________ Does that mean that method is correct? __No_____

Using the maps on the next page, try the easier method for a few minutes, and see what you get.

Try several different routes until you get a solution, or until you think a solution doesn't exist.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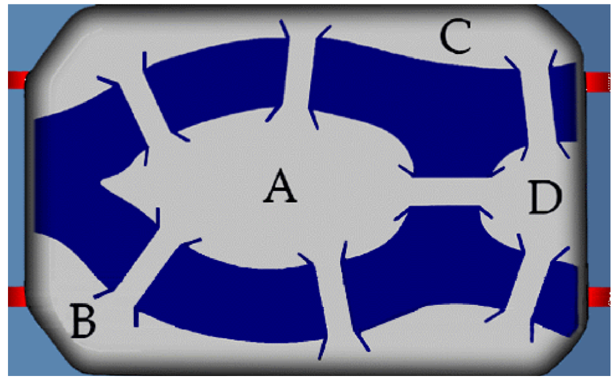
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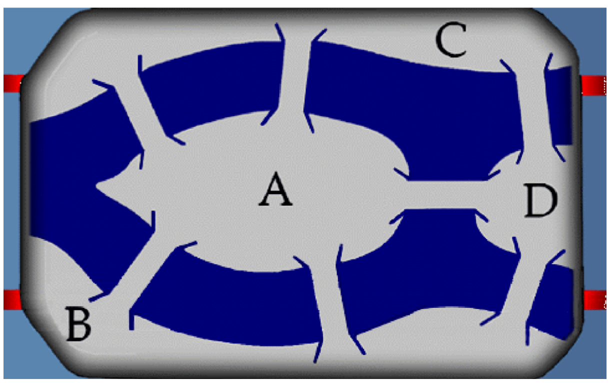
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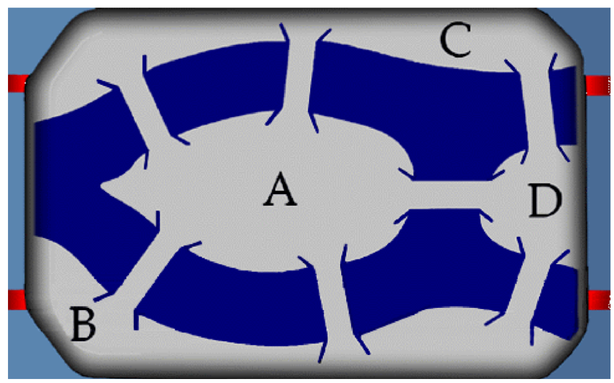
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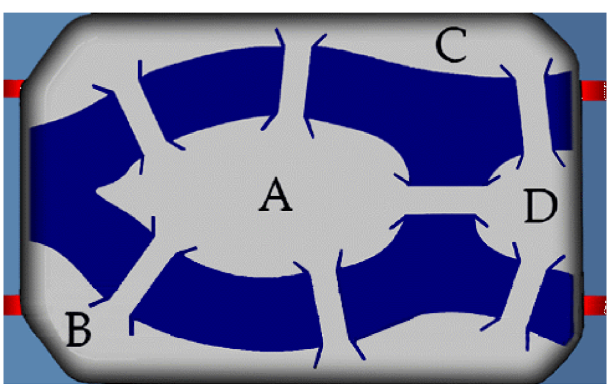
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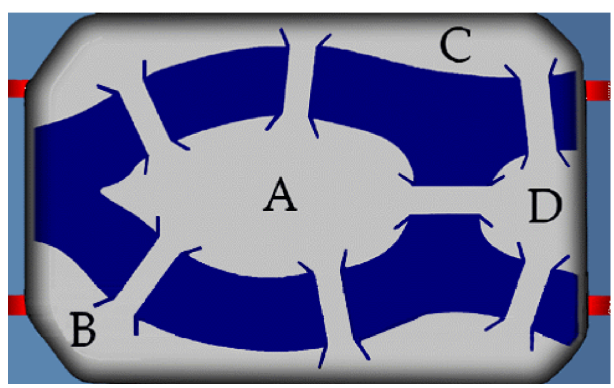
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Answer to Problem 1: No. (Euler, 1736)

To see this, we can view the map of Konigsburg as a *graph*. Each land mass is represented by a *vertex,* and each bridge is represented by an *edge*. A route through the graph that uses an edge at most one time is called a *trail.*

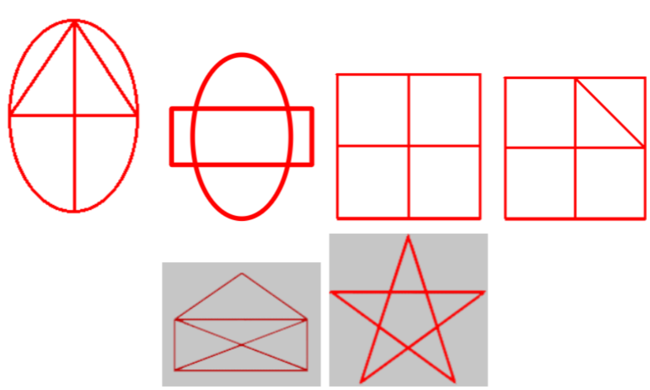
The steps of the *claim* and *proof* are posted below, but in the wrong order. Unscramble the steps to obtain a complete and correct claim and proof of the problem.

1. **Proof:**
2. Call this graph *G.* Hence, we see that *G* consists of _______ vertices and ________ edges.
3. **Claim:**
4. There is no route through Konigsburg that traverses every bridge exactly once.
5. As a result, there cannot exist a trail in *G* that contains every edge of *G.*
6. *Note* that each vertex in *G* touches an ________ number of edges.
7. It now suffices to prove the claim, "There is no *trail* in *G* that contains every *edge* of *G*."
8. However, each vertex in *G* touches an __________ number of edges.
9. Hence, if *T* contains every edge in *G*, then *T* must have at least two vertices that touch an even number of edges in *T*.
10. ☐
11. Except for possibly the beginning and ending vertices, every vertex in a trail *T* touches an (circle one) *even/odd* number of edges in *T*.
12. We represent the map of Konigsburg with a *graph* in the following way: draw a *vertex* for each land mass and an *edge* for each bridge.
13. This is because each middle vertex in *T* is entered by one edge and then exited by another.

Using the above reasoning, we can generalize the answer to Problem 1 as follows:

Proposition 2: _If a graph *G* has precisely zero or two vertices of odd degree, then there exists a trail containing every edge of *G.* ____.

Using Proposition 2, which of the following graphs might contain a trail with every edge? For those that cannot, can you succinctly explain why? For those that might, can you find such a trail?



Were there any graphs where you expected to find a trail with every edge but didn't? _______________
What does this suggest?

Conjecture 3: If within a graph *G* there exists a trail containing every edge in *G,* then *G* has precisely zero or two vertices of odd degree.____.

Combining Proposition 2 and Conjecture 3, can we make an even stronger conjecture?

Conjecture 4: A graph *G* has precisely zero or two vertices of odd degree *if and only if* there exists a trail containing every edge of *G.* ____.

We'll revisit this topic later in the semester…

Homework (due Monday): Writing Assignment #1